Structure-preserving numerical methods in relativity

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Advances and Challenges in Computational Relativity

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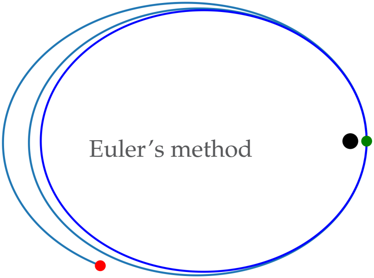
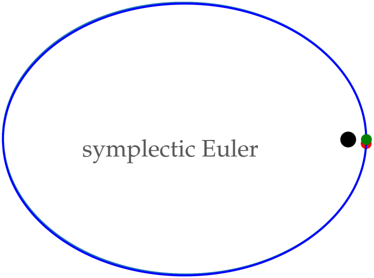
Structure-preserving discretization

Structure-preserving discretizations are numerical methods which preserve, on the discrete level, key geometric, topological, and algebraic structures possessed by the original continuous system.

Classical examples are symplectic integrators for Hamiltonian ODEs which preserve the symplectic 2-form associated to the Hamiltonian.
Here the Kepler problem is integrated over 4 periods using 200,000 timesteps.



$$\frac{x_{n+1} - x_n}{h} = v_n$$
$$\frac{v_{n+1} - v_n}{h} = -\frac{x_n}{|x_n|^3}$$



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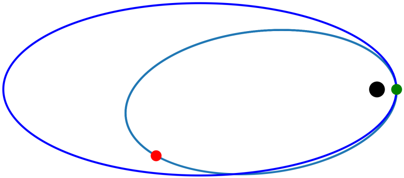
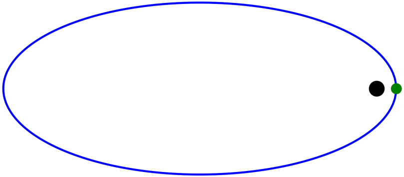
Higher order methods

1,000 periods, 1,000,000 time steps



Classical 4-stage Runge–Kutta



4-stage Gauss–Legendre (symplectic)

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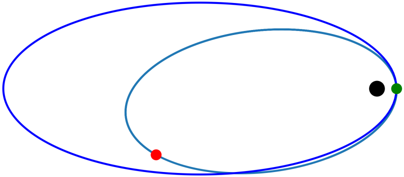

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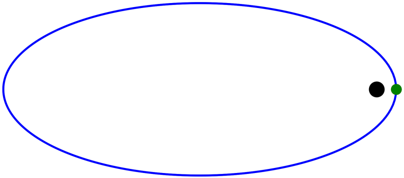
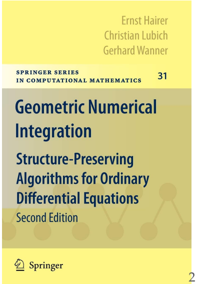


4-stage Gauss–Legendre
(symplectic)

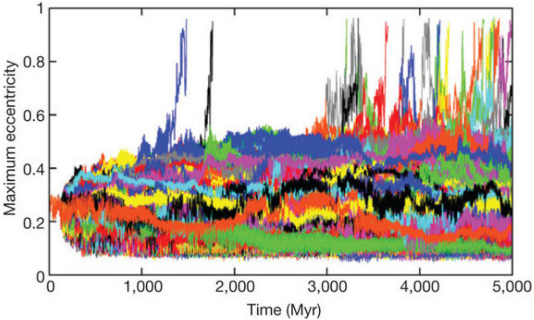
Ruth 1983, Feng Kang 1985, Sanz-Serna 1990, Leimkuhler-Reich 2004



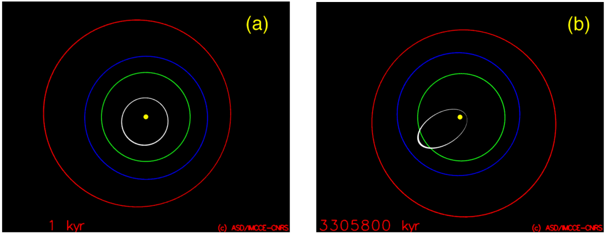
Symplectic integration of the solar system

In a famous 2009 paper in *Nature*, Laskar and Gastineau used a symplectic integrator to simulate the evolution of the solar system for the next 5 Gyr!

In fact, they did it 2,500 times, varying the initial position of Mercury by 0.38 mm each time.

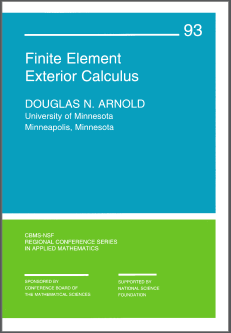


1% of the simulations resulted in unstable or collisional orbits.



Structure-preserving discretization for PDEs:

Finite Element Exterior Calculus



Hilbert complexes

FEEC applies to PDEs which are related to a *Hilbert complex*.

Hilbert space

closed, densely-defined operator

domain of d

$$\cdots \to W^{k-1} \xrightarrow{d^{k-1}, V^{k-1}} W^k \xrightarrow{d^k, V^k} W^{k+1} \to \cdots$$

The sequence must be a complex: $d \circ d = 0$

Further we require that the Hilbert complex is *closed*, meaning that the range of each d is closed. (This can be difficult to prove.)

Example:

$$\cdots \to L^2(\Omega) \xrightarrow{\mathrm{grad}, H^1(\Omega)} L^2(\Omega, \mathbb{R}^3) \xrightarrow{\mathrm{curl}, H(\mathrm{curl})} L^2(\Omega, \mathbb{R}^3) \to \cdots$$

Structure of Hilbert complexes

$$\cdots \to W^{k-1} \xrightarrow{d^{k-1}, V^{k-1}} W^k \xrightarrow{d^k, V^k} W^{k+1} \to \cdots$$

Each closed Hilbert complex has:

- cohomology and harmonic forms: $\mathcal{N}(d^k) / \mathcal{R}(d^{k-1}) \approx \mathfrak{H}^k$
- dual complex: $\cdots \leftarrow W^{k-1} \xleftarrow{d_k^*, V_k^*} W^k \xleftarrow{d_{k+1}^*, V_{k+1}^*} W^{k+1} \leftarrow \cdots$
- duality: $\mathcal{N}(d^k)^\perp = \mathcal{R}(d_{k+1}^*)$
- Hodge decomposition: $W^k = \mathcal{R}(d) \oplus \mathfrak{H} \oplus \mathcal{R}(d^*)$
- Poincaré inequality: $\|u\| \le c\|du\| \quad \forall u \perp \mathcal{N}(d)$
- and more …

The Hodge Laplacian

$$W^{k-1} \underset{d^*}{\overset{d}{\rightleftarrows}} W^k \underset{d^*}{\overset{d}{\rightleftarrows}} W^{k+1}$$

For every Hilbert complex and for each k, there is associated the *Hodge Laplacian* operator

$$dd^* + d^*d : W^k \to W^k$$

The Hodge-Laplace equation $(dd^* + d^*d)u = f$ is always well-posed up to harmonic forms = nullspace.

It has the obvious weak formulation

$$\langle d^*u, d^*v\rangle + \langle du, dv\rangle = \langle f, v\rangle \quad \forall v \in V \cap V^*.$$

The *mixed weak formulation* turns out to be better for discretization: Find $\quad (\sigma, u) \in V^{k-1} \times V^k \quad$ s.t.

$$\begin{aligned} \langle \sigma, \tau\rangle \ - \ \langle u, d\tau\rangle &= 0, & \tau &\in V^{k-1}, \\ \langle d\sigma, v\rangle \ + \ \langle du, dv\rangle &= \langle f, v\rangle, & v &\in V^k. \end{aligned}$$

The de Rham complex on a domain in 3D

$$0 \to L^2(\Omega) \xrightarrow{\text{grad}} L^2(\Omega, \mathbb{R}^3) \xrightarrow{\text{curl}} L^2(\Omega, \mathbb{R}^3) \xrightarrow{\text{div}} L^2(\Omega) \to 0$$

This is a special case of the de Rham complex on any manifold in n-D:

$$0 \to \Lambda^0(\Omega) \xrightarrow{d} \Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(\Omega) \to 0$$

It is a closed Hilbert complex for any bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary.

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In 3D:

- The 0-form Hodge Laplacian $= -\Delta$ (standard Laplacian)
- The 1-form Hodge Laplacian $= \operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div}$ (vector Laplacian)
- The 2-form Hodge Laplacian $= \operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div}$ with different boundary conditions and different weak formulation.
- The 3-form Hodge Laplacian $= -\Delta$ with the mixed weak formulation.

The Hodge wave equation

$$\ddot{U} + (dd^* + d^*d)U = 0, \quad U(0) = U_0, \quad \dot{U}(0) = U_1$$

Then $\sigma := d^*U,\ \rho := dU,\ u := \dot{U}$ satisfy

$$\begin{pmatrix} \dot\sigma \\ \dot u \\ \dot\rho \end{pmatrix} + \begin{pmatrix} 0 & -d^* & 0 \\ d & 0 & d^* \\ 0 & -d & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ u \\ \rho \end{pmatrix} = 0$$

strong

Find $\quad (\sigma, u, \rho) : [0,T] \to V^0 \times V^1 \times W^2 \quad$ s.t.

$$\begin{aligned} \langle \dot\sigma, \tau \rangle \ - \ \langle u, d\tau \rangle &= 0, && \tau \in V^0, \\ \langle \dot u, v \rangle \ + \ \langle d\sigma, v \rangle + \langle \rho, dv \rangle &= 0, && v \in V^1, \\ \langle \dot\rho, \eta \rangle \ - \ \langle du, \eta \rangle &= 0, && \eta \in W^2. \end{aligned}$$

weak

Theorem

For any initial data $(\sigma_0, u_0, \rho_0) \in V^0 \times V^1 \times W^2$, *there exists a unique solution* $(\sigma, u, \rho) \in C^0([0,T], V^0 \times V^1 \times W^2) \cap C^1([0,T], W^0 \times W^1 \times W^2)$.

Example: Maxwell's equations

$$\dot{D} = \operatorname{curl} H \qquad \dot{B} = -\operatorname{curl} E$$
$$\operatorname{div} D = 0 \qquad \operatorname{div} B = 0$$
$$D = \epsilon E \qquad B = \mu H$$

$$W^0 = L^2(\Omega)$$
$$W^1 = L^2(\Omega, \mathbb{R}^3, \epsilon\, dx)$$
$$W^2 = L^2(\Omega, \mathbb{R}^3, \mu^{-1} dx)$$

$$W^0 \xrightarrow{\operatorname{grad}, H^1} W^1 \xrightarrow{-\operatorname{curl}, H(\operatorname{curl})} W^2$$

Find $(\sigma, E, B) : [0, T] \to H^1 \times H(\operatorname{curl}) \times L^2$ s.t.

$$\langle \dot{\sigma}, \tau \rangle - \langle \epsilon E, \operatorname{grad} \tau \rangle = 0 \ \forall \tau,$$
$$\langle \epsilon \dot{E}, F \rangle + \langle \epsilon \operatorname{grad} \sigma, F \rangle - \langle \mu^{-1} B, \operatorname{curl} E \rangle = 0 \ \forall F,$$
$$\langle \mu^{-1} \dot{B}, C \rangle + \langle \mu^{-1} \operatorname{curl} E, C \rangle = 0 \ \forall C.$$

THEOREM

For any given initial data, there is a unique solution σ, E, and B to this Hodge wave equation. If the initial data satisfies the constraints (σ, $\operatorname{div} D$, and $\operatorname{div} B$ vanish at $t = 0$), and we set $D = \epsilon E$, $H = \mu^{-1} B$, then σ vanishes identically, and E, B, D, H satisfy Maxwell's equations.

Discretization of the Hodge Laplacian and Hodge wave eq

Discretize the mixed weak formulation by Galerkin's method using finite dimensional space $V_h^k \subset V^k$.

The discretization is stable and convergent under two crucial conditions:

- Subcomplex property: $dV_h^k \subset V_h^{k+1}$
- Bounded cochain projections:

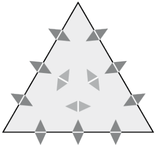
$$\begin{tikzcd} \cdots \arrow[r] & V^{k-1} \arrow[r, "d"] \arrow[d, "\pi_h^{k-1}"] & V^k \arrow[r, "d"] \arrow[d, "\pi_h^k"] & V^{k+1} \arrow[r] \arrow[d, "\pi_h^{k+1}"] & \cdots \\ \cdots \arrow[r] & V_h^{k-1} \arrow[r, "d"] & V_h^k \arrow[r, "d"] & V_h^{k+1} \arrow[r] & \cdots \end{tikzcd}$$

When these hold, the discrete spaces themselves form a Hilbert complex, and the discretization of the Hodge Laplacian is exactly the Hodge Laplacian associated to this discrete Hilbert complex. This leads to stability and convergence.
Structure-preserving discretization!

Finite element Galerkin subspaces

A key challenge is to construct finite element spaces to use as the V_h^k. This means specifying:

- element shape (simplex, cube, ...)
- space of shape functions on each element (scalar or vector polynomials up to a certain degree, perhaps trimmed or enriched ...)
- degrees of freedom (associated to faces, unisolvent)

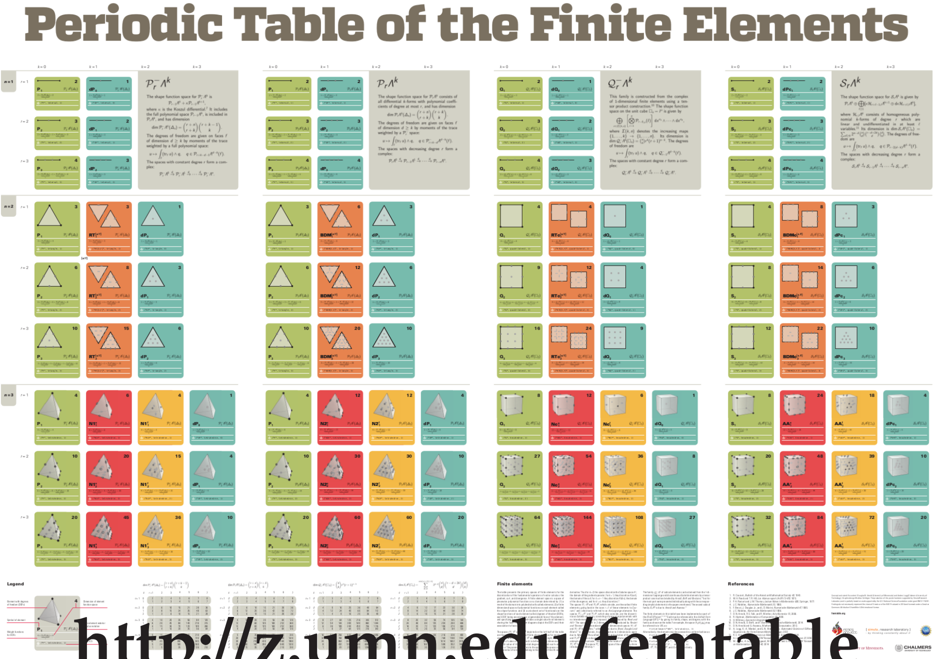


$\mathcal{P}_2\Lambda^1(\mathbb{R}^2)$

and ensuring the subcomplex and bounded cochain projection properties.

Remarks:

1. The degrees of freedom are crucial for cochain projections.
2. The construction of such structure-preserving finite element spaces is specific to each particular complex.
3. For the de Rham complex the FE spaces to use for each order of differential forms are well understood.



http://z.umn.edu/femtable

Motivating examples for FEEC

Eigenvalues of 1-form Laplacian

$$(d^*d + dd^*)u = (\operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div})u = \lambda u, \quad u \cdot n = 0, \operatorname{curl} u = 0 \text{ on bdry}$$

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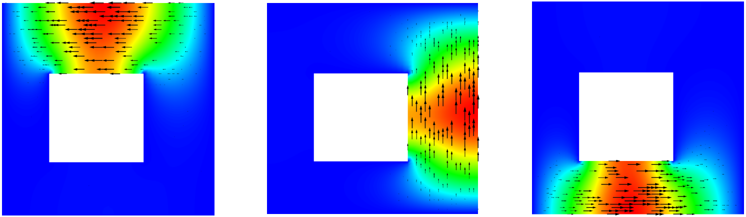
Find nonzero $u \in \Lambda^1$ such that $(du, dv) + (d^*u, d^*v) = \lambda(u, v) \quad \forall v \in \Lambda^1$



std $\mathcal{P}_1$ FEM $\quad \lambda_1 = 1.94 \quad \lambda_2 = 2.02 \quad \lambda_3 = 2.26$

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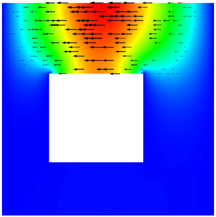
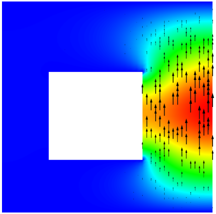
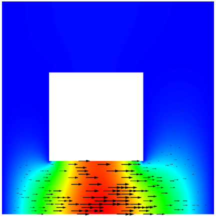

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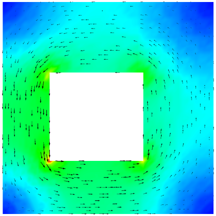
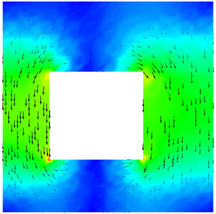
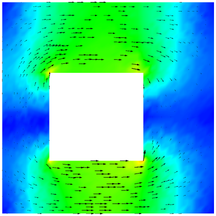
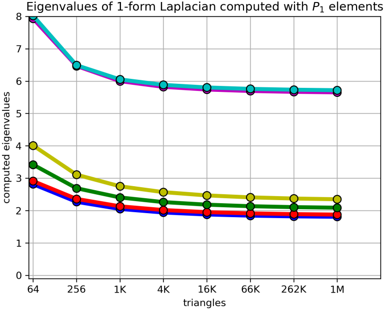




FEEC $\quad \lambda_1 = 0 \quad \lambda_1 = 0.617 \quad \lambda_2 = 0.658$

Convergence plots for the 1-form Laplacian, $\mathcal{P}_1$ vs FEEC

$\mathcal{P}_1$ finite elements



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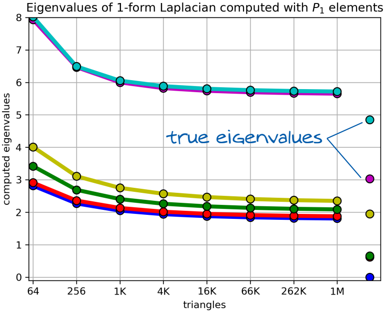

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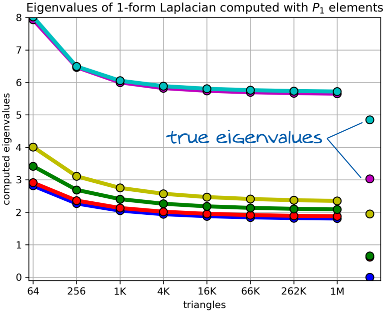
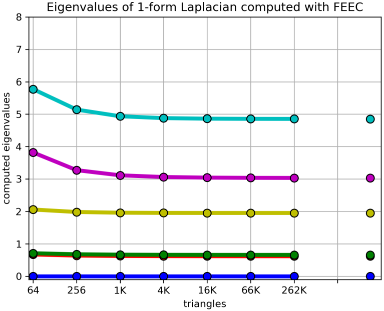


FEEC



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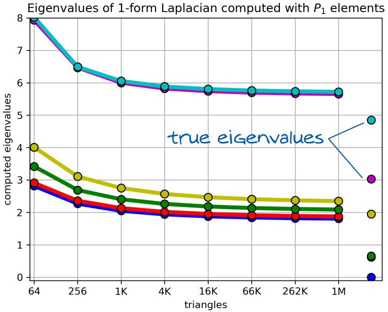

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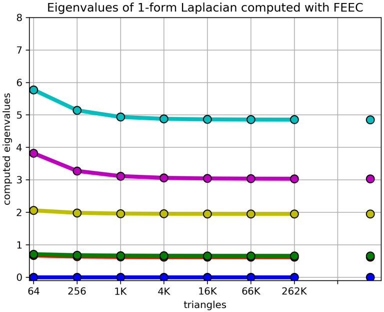

The standard $\mathcal{P}_1$ FEM is not convergent for this problem. Nightmare!

Fortunately, FEEC saves the day!

The Maxwell eigenvalue problem with std $\mathcal{P}_1$ FE

Find nonzero u such that $(du, dv) = \lambda(u, v) \quad \forall v$ (here $d = \text{curl}$)

Brezzi–Boffi–Gastaldi '99

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For $\Omega = (0, \pi) \times (0, \pi), \quad \lambda = m^2 + n^2, \quad m, n > 0$

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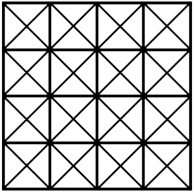
elts: 16	64	256	1024	4096
2.2606	2.0679	2.0171	2.0043	2.0011
4.8634	5.4030	5.1064	5.0267	5.0067
5.6530	5.4030	5.1064	5.0267	5.0067
5.6530	5.6798	5.9230	5.9807	5.9952
11.3480	9.0035	8.2715	8.0685	8.0171
11.3480	11.3921	10.4196	10.1067	10.0268
12.2376	11.4495	10.4197	10.1067	10.0268
12.2376	11.6980	13.7043	13.1804	13.0452
12.9691	11.6980	13.7043	13.1804	13.0452
13.9508	15.4308	13.9669	14.7166	14.9272



criss-cross meshes

Brezzi–Boffi–Gastaldi '99

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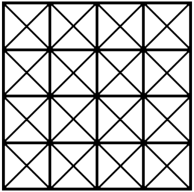

criss-cross meshes

Another nightmare (but with different origins).

Brezzi–Boffi–Gastaldi '99

The Maxwell eigenvalue problem with FEEC

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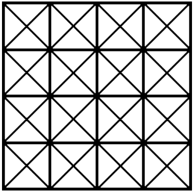
4.1577	4.8929	4.9749	4.9938	4.9985
4.1577	4.8929	4.9749	4.9938	4.9985
8.2543	7.4306	7.8619	7.9657	7.9914
9.7268	9.8498	9.9858	9.9975	9.9994
12.0419	9.8498	9.9858	9.9975	9.9994
12.0419	11.7309	12.7120	12.9292	12.9824
12.7326	11.7309	12.7120	12.9292	12.9824
14.5903	14.8468	17.0707	17.0240	17.0064
14.5903	15.3170	17.0707	17.0240	17.0064



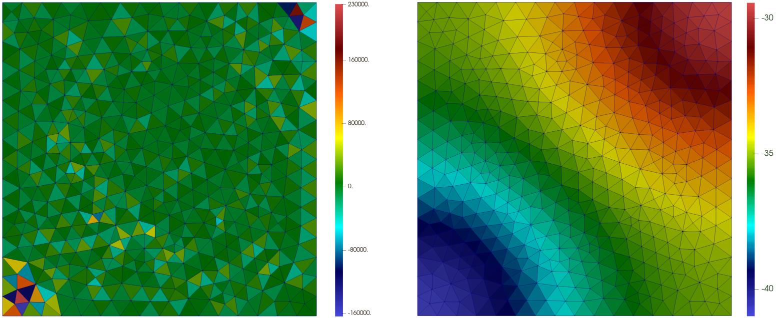
criss-cross meshes

Perfect!

Darcy flow, mixed formulation, std elts vs FEEC elts

$$v = -a \operatorname{grad} p, \quad \operatorname{div} v = f$$

$\mathcal{P}_1 \times \mathcal{P}_1$ $\mathcal{P}_0$ $\mathcal{P}_1^- \Lambda^1$ $\mathcal{P}_1^- \Lambda^2$



computed pressure fields

Raviart–Thomas 1975

Other complexes

The elasticity complex

$$0 \to L^2(\Omega, \mathbb{R}^3) \xrightarrow{\text{def}} L^2(\Omega, \mathbb{S}^3) \xrightarrow{\text{inc}} L^2(\Omega, \mathbb{S}^3) \xrightarrow{\text{div}} L^2(\Omega, \mathbb{R}^3) \to 0$$

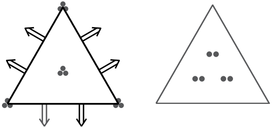
The differentials:

- $d^0 V = \operatorname{def} V = \text{symm grad}\, V = \mathcal{L}_V \delta_{ij}$ (Killing operator)
- $d^1 g = \operatorname{curl}(\operatorname{curl} g)^T$ (linearized Einstein curvature)
- $d^2 G = \operatorname{div} G$ (linearized contracted Bianchi identity)

The Hodge Laplacians:

- The 0 Hodge Lap. is the elasticity equations in terms of the displacement vector field.
- The 1 Hodge Lap. is a system for elastoplasticity taking into account crystal dislocations.
- The 2 Hodge Lap. is a different formulation of this.
- The 3 Hodge Lap. is the stress–displacement mixed formulation of elasticity.

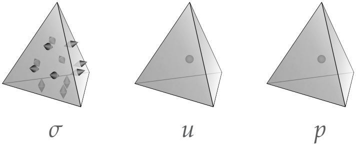
Since the work of Fraeijs de Veubeke in the 1960s, engineers and mathematicians have sought stable mixed finite elements for elasticity. With FEEC this problem was finally solved 40 years later (DNA-Winther 2002).



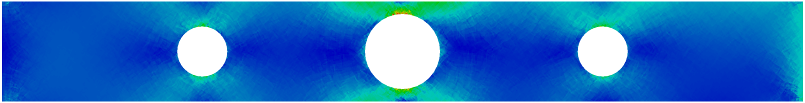
Lowest order AW elements for stress and displacement

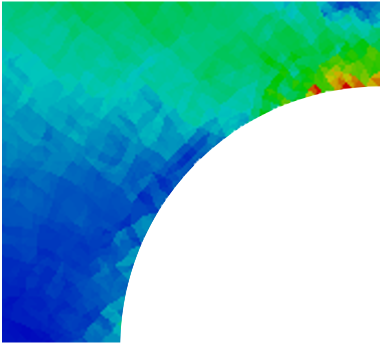
These elements have been quite successful in 2D. The 3D analogues have been developed, but are very complicated (Adams–Cockburn 2005, DNA–Awanou–Winther 2008, Hu–Zhang 2015).

This motivated a variant mixed formulation of elasticity in which the stress tensor is not required to be symmetric a priori, but an additional variable is introduced to enforce the symmetry. The new system fits in the FEEC framework and very simple elements have been found for it in n-dimensions (DNA–Falk–Winther 2007).

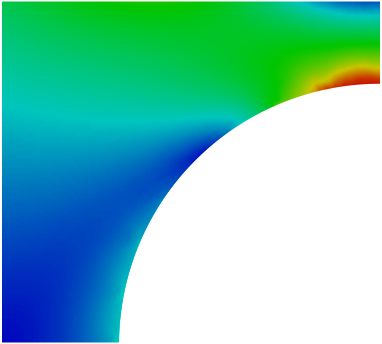


Variant elements: Cockburn, Gopalakrishnan, Guzmán, Stenberg, …





standard



FEEC

The Hessian complex

$$0 \to L^2(\Omega) \xrightarrow{\text{hess}} L^2(\Omega, \mathbb{S}^3) \xrightarrow{\text{curl}} L^2(\Omega, \mathbb{T}) \xrightarrow{\text{div}} L^2(\Omega, \mathbb{R}^3) \to 0$$

The differentials:

- $d^0 p = \operatorname{hess} p = \operatorname{grad} \operatorname{grad} p$ (Hessian)
- $d^1 g = \operatorname{curl} g$
- $d^2 G = \operatorname{div} G$

The 0 Hodge Lap. is Kirchhoff's plate equation (biharmonic).

The 1 Hodge Lap. is the Einstein–Bianchi formulation of GR.

The Einstein–Bianchi equations

On any pseudo-Riemannian manifold, the Riemann curvature tensor $\mathrm{Riem} = R_{abcd}$ captures the intrinsic curvature determined by the metric $g = g_{ab}$. It satisfies the symmetries:

$$R_{(ab)cd} = R_{ab(cd)} = 0, \qquad R_{abcd} = R_{cdab}, \qquad R_{[abcd]} = 0$$

antisymm. by pairs — interchange symm. — null totally antisymm. part

In 4D this reduces the 256 components to 20 independent ones.

Bianchi identity: $\nabla_{[a} R_{bc]de} = 0$

The Ricci tensor is the trace, $\mathrm{Ric} = \mathrm{tr}\,\mathrm{Riem}$: $R_{bd} = g^{ac} R_{abcd}$ (10 comps)

The vacuum Einstein equations are $\mathrm{Ric} = 0.$

Weyl curvature

Ric contributes 10 of the 20 components of Riem. The remainder is the Weyl tensor:

$$\mathrm{Riem} = \mathrm{Ric} + \mathrm{Weyl}$$

$$R_{abcd} = \overbrace{\left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a} - \frac{1}{3}g^{ef}R_{ef}g_{a[c}g_{d]b}\right)}^{\text{portion of Riem reconstructed from Ric}} + C_{abcd}$$

Weyl satisfies the symmetries of Riem and is also trace-free:

$$g^{ac}C_{abcd} = 0$$

Vacuum Einstein $\iff$ $\mathrm{Riem} = \mathrm{Weyl}$

Einstein–Bianchi equations

Friedrich '96; Anderson–Choquet-Bruhat–York '97

Contracting the Bianchi identity gives

$$\nabla^a R_{abcd} = \nabla_d R_{bc} - \nabla_c R_{bd}$$

If the vacuum Einstein equations hold, $R_{abcd} = C_{abcd}$ and $R_{ab} = 0$, so

$$\nabla^a C_{abcd} = 0.$$

The Einstein–Bianchi system uses these equations to determine Weyl.

The Bel decomposition (linear case)

Bel '58

Let C_{abcd} denote the linearized Weyl tensor. Because of antisymmetry by pairs, it is determined by the components C_{abcd} with $ab, cd \in \{01, 02, 03, 23, 31, 12\}$.

$$\begin{array}{ccc}
 & \begin{array}{cc} \text{time} \qquad\qquad\qquad & \text{space} \end{array} & \\
(C_{abcd}) = & \left(\begin{array}{ccc|ccc}
C_{0101} & C_{0102} & C_{0103} & C_{0123} & C_{0131} & C_{0112} \\
C_{0201} & C_{0202} & C_{0203} & C_{0223} & C_{0231} & C_{0212} \\
C_{0301} & C_{0302} & C_{0303} & C_{0323} & C_{0331} & C_{0312} \\
\hline
C_{2301} & C_{2302} & C_{2303} & C_{2323} & C_{2331} & C_{2312} \\
C_{3101} & C_{3102} & C_{3103} & C_{3123} & C_{3131} & C_{3112} \\
C_{1201} & C_{1202} & C_{1203} & C_{1223} & C_{1231} & C_{1212}
\end{array}\right) & \begin{array}{c} \text{time} \\ \\ \text{space} \end{array} =: \begin{pmatrix} E & B \\ H & D \end{pmatrix}
\end{array}$$

$E, B, H, D \in \mathbb{R}^{3\times 3}$.

Cf. the Faraday electromagnetic 2-form

$$(F_{ab}) = \left(\begin{array}{c|ccc}
0 & E_1 & E_2 & E_3 \\
\hline
-E_1 & 0 & B_3 & -B_2 \\
-E_2 & -B_3 & 0 & B_1 \\
-E_3 & B_2 & -B_1 & 0
\end{array}\right), \qquad F = dt \wedge E + B$$

Weyl symmetries in terms of the Bel decomposition

Theorem

A fourth order tensor

$$C_{abcd} = \begin{pmatrix} E & B \\ H & D \end{pmatrix}$$

satisfies the symmetries of the Weyl tensor if and only if $H = B$, $D = -E$, and E and B are symmetric and trace-free, i.e.,

$$(C_{abcd}) = \begin{pmatrix} E & B \\ B & -E \end{pmatrix}, \quad E, B \text{ symmetric, trace-free}$$

Linearized Einstein–Bianchi in terms of the Bel decomposition

For each $b \in 0, 1, 2, 3$, the EB equations $\nabla^a C_{abcd} = 0$ gives 6 PDEs.

$b = 0$:	$\operatorname{div} E = 0$, $\operatorname{div} B = 0$	*6 constraint equations*
$b = 1, 2, 3$:	$\dot{E} = -\operatorname{curl} B$, $\dot{B} = \operatorname{curl} E$	*18 evolution equations*

For any initial data, the unconstrained evolution equations are well-posed.

Constraint propagation follows from:

Proposition

If $E : \Omega \to \mathbb{R}^{3\times3}$ is trace-free, symmetric, and divergence-free (TSD), then so is $\operatorname{curl} E$.

The proof is based on the identities collected in the following diagram:

$$
\begin{tikzcd}
L^2 \arrow[r, "\operatorname{grad}"] & L^2 \otimes \mathbb{R}^3 \arrow[r, "\operatorname{curl}"] & L^2 \otimes \mathbb{R}^3 \arrow[r, "\operatorname{div}"] & L^2 \\
L^2 \otimes \mathbb{R}^3 \arrow[r, "\operatorname{grad}"] \arrow[ur, "\operatorname{id}"] & L^2 \otimes \mathbb{R}^{3\times 3} \arrow[r, "\operatorname{curl}"] \arrow[ur, "-\operatorname{skw}"] & L^2 \otimes \mathbb{R}^{3\times 3} \arrow[r, "\operatorname{div}"] \arrow[ur, "\operatorname{tr}"] & L^2 \otimes \mathbb{R}^3 \\
L^2 \otimes \mathbb{R}^3 \arrow[r, "\operatorname{grad}"] \arrow[ur, "\operatorname{inc}"] & L^2 \otimes \mathbb{R}^{3\times 3} \arrow[r, "\operatorname{curl}"] \arrow[ur, "S"] & L^2 \otimes \mathbb{R}^{3\times 3} \arrow[r, "\operatorname{div}"] \arrow[ur, "-\operatorname{skw}"] & L^2 \otimes \mathbb{R}^3 \\
L^2 \arrow[r, "\operatorname{grad}"] \arrow[ur, "I"] & L^2 \otimes \mathbb{R}^3 \arrow[r, "\operatorname{curl}"] \arrow[ur, "\operatorname{inc}"] & L^2 \otimes \mathbb{R}^3 \arrow[r, "\operatorname{div}"] \arrow[ur, "\operatorname{id}"] & L^2
\end{tikzcd}
$$

$Sm = m^T - (\operatorname{tr} m)I$

Einstein–Bianchi is the 1 Hodge wave eq. for Hessian complex

$$L^2(\Omega) \xrightarrow{\text{hess}, H^2} L^2(\Omega;\mathbb{S}) \xrightarrow{\text{curl}, H(\text{curl})} L^2(\Omega;\mathbb{T})$$

Find $\quad (\sigma, E, B) : [0,T] \to H^2 \times H(\text{curl};\mathbb{S}) \times L^2(\Omega;\mathbb{T}) \quad$ s.t.

$$\begin{aligned} &\langle \dot\sigma, \tau\rangle - \langle u, \text{hess}\,\tau\rangle = 0, && \tau \in H^2,\\ &\langle \dot E, F\rangle + \langle \text{hess}\,\sigma, F\rangle + \langle B, \text{curl}\,F\rangle = 0, && F \in H(\text{curl};\mathbb{S}),\\ &\langle \dot B, C\rangle - \langle \text{curl}\,E, C\rangle = 0, && C \in L^2(\Omega;\mathbb{T}). \end{aligned}$$

$$\dot\sigma = \text{div}\,\text{div}\,E, \quad \dot E = -\,\text{hess}\,\sigma - \text{sym}\,\text{curl}\,B, \quad \dot B = \text{curl}\,E$$

THEOREM

Suppose $\sigma(0) = 0$ and $E(0)$ and $B(0)$ are TSD. Then $\sigma = 0$ and E and B are TSD for all time, and E and B satisfy the linearized EB equations.

Quenneville-Belair, PhD thesis 2015

Obstacles to discretization

To proceed we need finite element subspaces which form a subcomplex with bounded cochain projections. There are two serious obstacles.

1. It is difficult to create a finite element subspace of H^2 because of the second derivatives. A similar situation arose in FEEC methods for the plate equation.

2. It is difficult to create a finite element subspace of $H(\mathrm{curl}; \mathbb{S})$ because of the symmetry. A similar situation arose in FEEC for elasticity.

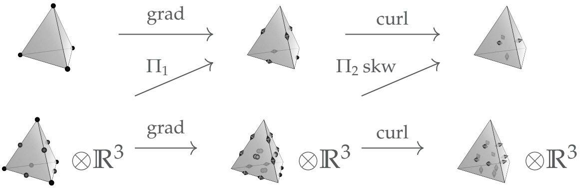
V. Quenneville-Belair took the approach of introducing new variables (weak symmetry etc.) and arrived at a discretization with six fields using low order elements. Recently Hu–Liang discretized the Hessian complex directly, but the resulting elements are high order and very complicated.

The FEEC formulation of the EB system

Combining ideas leads to a 1st order formulation of EB using 6 variables.

$$
\begin{tikzcd}
\sigma & & \\
L^2(\Omega) \arrow[r, "\mathrm{grad}"] & L^2(\Omega;\mathbb{R}^3) \arrow[r, "\mathrm{curl}"] & L^2(\Omega;\mathbb{R}^3) \\
L^2(\Omega;\mathbb{R}^3) \arrow[r, "\mathrm{grad}"] \arrow[ur, "I"] & L^2(\Omega;\mathbb{R}^{3\times 3}) \arrow[r, "\mathrm{curl}"] \arrow[ur, "\mathrm{skw}"] & L^2(\Omega;\mathbb{R}^{3\times 3}) \\
& E & B
\end{tikzcd}
$$

FEEC then guides us to a stable choice of elements.



Combining ideas leads to a 1st order formulation of EB using 6 variables.

$$\begin{tikzcd}
\overset{\sigma}{L^2(\Omega)} \arrow[r, "\mathrm{grad}"] & L^2(\Omega;\mathbb{R}^3) \arrow[r, "\mathrm{curl}"] & L^2(\Omega;\mathbb{R}^3) \\
L^2(\Omega;\mathbb{R}^3) \arrow[ur, "I"] \arrow[r, "\mathrm{grad}"] & \underset{E}{L^2(\Omega;\mathbb{R}^{3\times 3})} \arrow[ur, "\mathrm{skw}"] \arrow[r, "\mathrm{curl}"] & \underset{B}{L^2(\Omega;\mathbb{R}^{3\times 3})}
\end{tikzcd}$$

FEEC then guides us to a stable choice of elements.